Calculus: The Mathematics of Change

Course Description

Students study limits, differentiation, integration, and applications of both differentiation and integration including but not limited to: analysis of real-valued functions of a single variable represented graphically, numerically, analytically, and verbally; related rates; curve sketching; optimization; solving separable differential equations; length, area, volume, and work.

Textbook

Larson, Hostetler, & Edwards. *Calculus of a Single Variable, 8th Ed.* Belmont, CA: Brooks/Cole Cengage Learning, 2006.

Required Materials

Pencil, ruler, and eraser
Graphing calculator: Texas Instruments TI84
Loose-leaf (college-ruled or quadrille) paper for note taking and completion of homework
3-ring binder or a designated section of a 3-ring binder for storing notes, homework, quizzes, and tests

Grading System

Each semester will include four or more units of study. Each unit will include the creation of a *unit packet* and will conclude in a formal end-of-unit assessment (test). Unit packets will be composed of notes taken by students, class handouts, and annotated homework assignments. Packets will be turned in on the day of the corresponding unit test.

Each semester students will earn a term average determined by scores earned in two weighted categories. Scores on unit packets and any other assessment of classroom *participation* will be weighted at 10% of the term average. Scores on formal end-of-unit assessments and occasional quizzes will be weighted at 90% of the term average.

An overall semester grade is determined by a student's term average, weighted at 80%, and cumulative exam score, weighted at 20%. Seniors may exempt the second semester cumulative exam if their term average prior to the exam is the equivalent of an "A" or "A-". If a senior exempts the second semester cumulative exam, then his/her overall second semester grade is determined solely by the term average.

An overall course grade is determined by calculating the average of the overall first semester grade and overall second semester grade.

Units of Study

Semester One

Limits and Their Properties (SEP 15)
Continuity and Asymptotes (OCT 1)
Differentiation (OCT 15)
Related Rates (NOV 1)
Curve Sketching (NOV 15)
Optimization (DEC 1)

Semester Two

Indefinite Integrals (FEB 1)
Definite Integrals (FEB 15)
Logarithmic and Exponential Functions (MAR 1)
Inverse Trigonometric Functions (MAR 15)
Differential Equations (APR 1)
Length, Area, Volume and Work (APR 15)

CONTENT OUTLINE WITH *ENDURING UNDERSTANDINGS & ESSENTIAL QUESTIONS*

LHE: Corresponding section in primary course textbook
AB: Indicates inclusion of the topic in the *College Board* course description for AP Calculus AB
BC: Indicates inclusion of the topic in the *College Board* course description for AP Calculus BC

SEMESTER ONE

1. Unit One: Limits and Their Properties (LHE 1.0)

    a. A Preview of Calculus (LHE 1.1)

        i. Analysis of one-dimensional motion of object given corresponding numerical, graphical, or algebraic representation of a linear or quadratic velocity function (AB; BC)

            1. Slope as velocity
            2. Area bound by the curve as displacement

        ii. Tangent line to a curve at a point and local linear approximation (AB; BC)

    b. Finding Limits Graphically and Numerically (LHE 1.2; AB; BC)

    c. Evaluating Limits Analytically (LHE 1.3; AB; BC)

        i. Calculating limits using algebra: factoring, combining rational expressions, multiplying by a conjugate (AB; BC)

    d. *Enduring Understandings* and *Essential Questions* for Unit One

        i. EU.1.1: The characteristics of a continuous curve can be approximated using secant lines over small intervals. Decreasing the size of the interval often increases the accuracy of the approximation.

            1. EQ.1.1.A: How can lines be used to approximate the behavior and/or characteristics of curves?

            2. EQ.1.1.B: How do you increase the accuracy of a linear approximation?

            3. EQ.1.1.C: Under what circumstances might a linear approximation be grossly inaccurate?

        ii. EU.1.2: The area of a bound region can be approximated using rectangles. Increasing the number of rectangles used to make the approximation almost always increases the accuracy of the approximation.

            1. EQ.1.2.A: How do you determine the area of a bound region?

            2. EQ.1.2.B: Can the exact area of a bound region always be determined? Why or why not?

        iii. EU.1.3: If an object changes position you can be certain that for some interval of time the object is in motion. Knowing both the coordinates of the initial position and final position of the object as well as the length of the time interval enables you to determine the average velocity of the object but not the instantaneous velocity of the object at a particular moment within the time interval.

            1. EQ.1.3.A: How can you be certain that an object is in motion?

            2. EQ.1.3.B: What must you know in order to determine the velocity (speed and direction) of an object?

3. EQ.1.3.C: What is the difference between average velocity and instantaneous velocity?

iv. EU.1.4: A limit is a value that one approaches and expects yet does not always realize. Limits do not always exist but when they do they can often be determined using graphical, numerical, and algebraic methods.

1. EQ.1.4.A: What is a limit?

2. EQ.1.4.B: What are the different ways in which a limit can be calculated?

3. EQ.1.4.C: What kind of algebraic maneuvers are most effective when evaluating limits?

2. Unit Two: Continuity and Asymptotes (LHE 1.0)

a. Continuity and One-Sided Limits (LHE 1.4)

i. An intuitive understanding of continuity where the function values can be made as close as desired by taking sufficiently close values of the domain (AB; BC)

ii. Understanding continuity in terms of limits (AB; BC)

iii. Intermediate Value Theorem (AB; BC)

b. Infinite Limits (LHE 1.5)

c. *Enduring Understandings* and *Essential Questions* for Unit Two

i. EU.2.1: For all continuous functions the *expected* value of the output as an input approaches any particular value in the domain is equal to the *actual* value of the output at that exact input value. Any continuous portion of a curve can be drawn by hand without lifting the writing instrument from the paper

1. EQ.2.1.A: How do you determine whether or not a function is continuous at a particular value of x in its domain? Over a particular interval of its domain?

2. EQ.2.1.B: Under what circumstances is a piecewise function continuous?

ii. EU.2.2: *Intermediate Value Theorem*: If a function $f(x)$ is continuous over the closed interval [a,b] then for any value C between $f(a)$ and $f(b)$ there exists at least one value of x in the open interval (a,b) such that $f(x)=C$.

1. EQ.2.2.A: What is the *Intermediate Value Theorem*?

2. EQ.2.2.B: Why does application of the *Intermediate Value Theorem* require the function to be continuous?

iii. EU.2.3: The end behavior of a function can be discerned by determining the limit of the function's output as the input increases toward infinity as well as the limit of the function's output as the input decreases toward negative infinity. When the output converges upon a particular value the end-behavior of the function can be described as "asymptotic."

1. EQ.2.3.A: How do you determine the end-behavior of a function?

2. EQ.2.3.B: What is a horizontal asymptote?

iv. EU.2.4: As a vertical asymptote is approached (from either side) the output of the function increases toward positive infinity and/or decreases toward negative infinity.

1. EQ.2.4.A: What does it mean to say that a function behaves "asymptotically"?

            2. EQ.2.4.B: Can the graph of a function pass through an asymptote?

3. Unit Three: Differentiation (LHE 2.0)

    a. The Derivative and the Tangent Line Problem (LHE 2.1; AB; BC)

        i. Slope of a curve at a point. Examples are emphasized, including points at which there are vertical tangents and points at which there are no tangents (AB; BC)

        ii. Derivative defined as the limit of the difference quotient (AB; BC)

        iii. Relationship between differentiability and continuity (AB; BC)

        iv. Instantaneous rate of change as the limit of average rate of change (AB; BC)

    b. Basic Differentiation Rules and Rates of Change (LHE 2.2; AB; BC)

        i. The Constant Rule

        ii. The Power Rule (AB; BC)

        iii. The Constant Multiple Rule

        iv. The Sum and Difference Rules

        v. Derivatives of Sine and Cosine Functions (AB; BC)

    c. Product and Quotient Rules and Higher-Order Derivatives (LHE 2.3; AB; BC)

        i. Interpretation of the derivative as a rate of change in varied applied contexts, including velocity, speed, and acceleration (AB; BC)

    d. *Enduring Understandings* and *Essential Questions* for Unit Three

        i. EU.3.1: The derivative of a function at a particular point is equal to the slope of the line tangent to the curve at that point. This slope can be determined by evaluating a limit of a difference quotient representing the slope of a secant line over an infinitely small interval.

            1. EQ.3.1.A: How are derivatives and tangent lines related?

        ii. EU.3.2: An instantaneous rate of change is the limit of an average rate of change over an infinitely small interval.

            1. EQ.3.2.A: What is the difference between an *average* rate of change and an *instantaneous* rate of change?

        iii. EU.3.3: Rules of differentiation can be discovered through the application of the definition of the derivative to a set of functions with similar characteristics. These rules allow the mathematician to determine derivative functions quickly without much need for algebraic manipulations.

            1. EQ.3.3.A: What is the definition of the derivative?

            2. EQ.3.3.B: How can one use the definition of the derivative to discover rules of differentiation?

        iv. EU.3.4: The derivatives of the sine function and cosine function are most easily discerned by considering slopes of tangents to the graphs of these functions. These derivatives exemplify the fact that the derivative of a periodic function is also periodic.

            1. EQ.3.4.A: Why must the derivative of a periodic function also be periodic?

4. Unit Four: Related Rates (LHE 2.0)

    a. The Chain Rule (LHE 2.4; AB; BC)

    b. Implicit Differentiation (LHE 2.5; AB; BC)

    c. Related Rates (LHE 2.6; AB; BC)

    d. *Enduring Understandings* and *Essential Questions* for Unit Four

        i. EU.4.1: The *Chain Rule* enables you to determine the derivative of a composite function without having to manipulate the form of the function.

            1. EQ.4.1.A: How does the chain rule allow you to determine derivatives more quickly?

            2. EQ.4.1.B: What alternative strategies can be used in order to determine a derivative without applying the chain rule?

        ii. EU.4.2: The technique of *implicit differentiation* enables you to differentiate a relationship between two variables without having to isolate a single variable.

            1. EQ.4.2.A: Why is it unnecessary to isolate a single variable (express one variable exclusively in terms of another) when differentiating implicitly?

            2. EQ.4.2.B: Can all algebraic relationships expressed in the form of a single equation be differentiated implicitly?

        iii. EU.4.3: Implicit differentiation (with respect to time) of an algebraic expression containing multiple variables yields an algebraic expression showing the relationship between the variables and the rates at which they are changing.

            1. EQ.4.3.A: How do you best describe a relationship between multiple variables with a single equation when the variables themselves are changing with respect to time?

5. Unit Five: Curve Sketching (LHE 3.0)

    a. Extrema on an Interval (LHE 3.1)

        i. Extreme Value Theorem (AB; BC)

    b. Rolle's Theorem and the Mean Value Theorem (LHE 3.2; AB; BC)

    c. Increasing and Decreasing Functions and the First Derivative Test (LHE 3.3)

    d. Concavity and the Second Derivative Test (LHE 3.4; AB; BC)

        i. Points of inflection as places where concavity changes (AB; BC)

    e. Limits at Infinity (LHE 3.5)

    f. A Summary of Curve Sketching (LHE 3.6)

        i. Analysis of curves, including the notions of monotonicity and concavity (AB; BC)

        ii. Corresponding characteristics of the graphs of f, f', and f''

        iii. Understanding asymptotes in terms of graphical behavior (AB; BC)

        iv. Describing asymptotic behavior in terms of limits involving infinity (AB; BC)

    g. *Enduring Understandings* and *Essential Questions* for Unit Five

i. EU.5.1: The secant line passing through two points on a continuous and differentiable function has a slope that is equal to the slope of at least one tangent line to the curve between the two points of intersection (Mean Value Theorem). When the secant line is horizontal one can be certain that a relative extreme exists between the two points of intersection (Rolle's Theorem).

    1. EQ.5.1.A: What is the Mean Value Theorem?

    2. EQ.5.1.B: How can the Mean Value Theorem be restated in terms of average velocity and instantaneous velocity?

    3. EQ.5.1.C: What is Rolle's Theorem?

    4. EQ.5.1.D: Does application of Rolle's Theorem require the function to be both continuous *and* differentiable? Why or why not?

ii. EU.5.2: The graph and/or values of the first derivative of a function reveal whether or not the function is increasing or decreasing as well as the location of relative (local) extremes.

    1. EQ.5.2.A: What can we learn about a function through an examination of its first derivative?

    2. EQ.5.3.A: How do you determine whether or not a relative extreme is a relative maximum or a relative minimum?

iii. EU.5.3: The graph and/or values of the second derivative of a function reveal the concavity of the function as well as the location of points of inflection.

    1. EQ.5.3.A: What can we learn about a function through an examination of its second derivative?

    2. EQ.5.3.B: What is concavity? What does it mean for a function to be concave up versus concave down?

6. Unit Six: Optimization (LHE 3.0)

    a. Optimization Problems (LHE 3.7; AB; BC)

        i. Absolute (global) and relative (local) extremes (AB; BC)

    b. *If Time:* Newton's Method (LHE 3.8)

    c. *If Time:* Differentials (LHE 3.9)

    d. *Enduring Understandings* and *Essential Questions* for Unit Six

        i. EU.6.1: When a relation or function can be written to model the relationship between variables, differentiation can be used to determine the optimal conditions under which a particular variable is maximized or minimized. This is particularly useful when the variable to be minimized is materials, cost, or time or when the variable to be maximized is area, volume, or profit.

            1. EQ.6.1.A: How can calculus be used to determine optimal conditions?

        ii. EU.6.2: A zero of a function can be approximated by considering the zeros of linear approximations (tangent lines) at points converging upon the point at which the sign of the function changes.

            1. EQ.6.2.A: How can calculus be used to approximate the location of an x-intercept?

iii. EU.6.3: A differential represents an infinitely small change in the value of a variable. Although the differential's value is infinitely small it is not equal to zero and therefore behaves and can be manipulated like any other nonzero real number.

1. EQ.6.3.A: What is a differential? How are differentials like zero? Unlike zero?

Semester Two

7. Unit Seven: Indefinite Integrals (LHE 4.0)

a. Antiderivatives and Indefinite Integration (LHE 4.1; AB; BC)

i. Motion along a line; distance traveled by a particle along a line (AB; BC)

ii. Finding specific antiderivatives using initial conditions (AB; BC)

iii. Determining accumulated change from a given rate of change (AB; BC)

b. Particular solutions to differential equations given an initial condition (LHE 4.1)

c. *Enduring Understandings* and *Essential Questions* for Unit Seven

i. EU.7.1: The displacement (change in position) of a particle is not always equal to the total distance traveled by the particle over a certain time interval. If the particle ever changes direction then the total distance traveled by the particle will be greater than the displacement of the particle.

1. EQ.7.1A: Why might the displacement of a particle not be equal to the total distance traveled by a particle?

ii. EU.7.2: There exist an infinite number of different functions that all have the same derivative function. Therefore when a first derivative function is given alone it is impossible to know the function from which it was derived without additional information.

1. EQ.7.2.A: Is it possible to determine the formula for a function given only the formula for its first derivative? Why or why not?

8. Unit Eight: Definite Integrals (LHE 4.0)

a. Area (LHE 4.2)

b. Riemann Sums and Definite Integrals (LHE 4.3; AB; BC)

i. Definite integral as a limit of Riemann sums (AB; BC)

ii. Basic properties of definite integrals: additivity and linearity (AB; BC)

c. The Fundamental Theorem of Calculus (LHE 4.4)

i. PROOF: The Fundamental Theorem of Calculus
ii. Use of the FTC to evaluate definite integrals (AB; BC)

iii. Use of the FTC to represent a particular antiderivative, and the analytical and graphical analysis of functions so defined (AB; BC)

iv. Average Value of a Function (AB; BC)

v. Definite integral of the rate of change of a quantity over an interval interpreted as the change of the quantity over the interval (AB; BC)

d. Integration by Substitution (LHE 4.5; AB; BC)

e. Numerical Integration (LHE 4.6)

f. *Enduring Understandings* and *Essential Questions* for Unit Eight

i. EU.8.1: A *Riemann Sum* is a sum of products of function outputs and small changes in the function's input. Although these products are represented graphically as areas of rectangles, these "areas" can take on both positive and negative values.

1. EQ.8.1.A: What is a Riemann Sum? Under what circumstances will a Riemann Sum be positive? Be negative?

ii. EU.8.2: The *Fundamental Theorem of Calculus* describes the inverse relationship between differentiation and integration. The method for evaluating a definite integral is corollary to this theorem.

1. EQ.8.2.A: How can the Fundamental Theorem of Calculus be used to justify the process for evaluating a definite integral?

iii. EU.8.3: The definite integral of a rate of change of a variable over an interval is equal to the net change in the value of the variable over the interval.

1. EQ.8.3.A: What might the definite integral of a rate of change represent?

iv. EU.8.4: Anticipating application of the *chain rule* enables you to determine the most effective substitution to be made into an integrand so that the antiderivative can more easily be determined.

1. EQ.8.4.A: Why might a thorough understanding of the *Chain Rule* enable one to make strategic substitutions when determining antiderivatives?

v. EU.8.5: The *fnInt* function of the TI84 graphing calculator enables users to obtain a quick approximation, and sometimes even an exact calculation, of the value of a definite integral. The calculator makes this approximation by calculating a Riemann sum with very small subinterval lengths.

1. EQ.8.5.A: How does the *fnInt* feature of the TI84 work? Can the calculator ever generate more than just a good approximation of a definite integral?

2. EQ.8.5.B: What factor is the accuracy of the calculator's approximation of a Riemann Sum most dependent upon?

9. Unit Nine: Logarithmic and Exponential Functions (LHE 5.0)

a. The Natural Logarithmic Function: Differentiation (LHE 5.1; AB; BC)

b. The Natural Logarithmic Function: Integration (LHE 5.2)

c. *If time:* Inverse Functions (LHE 5.3; AB; BC)

i. Use of implicit differentiation to find the derivative of an inverse function (AB; BC)

d. Exponential Functions: Differentiation and Integration (LHE 5.4; AB; BC)

i. Comparing relative magnitudes of functions and their rates of change: exponential versus polynomial versus logarithmic growth (AB; BC)

e. Bases Other Than *e* and Applications (LHE 5.5)

f. *Enduring Understandings* and *Essential Questions* for Unit Nine

i. EU.9.1: Logarithmic differentiation takes advantage of properties of logarithms and the technique of implicit differentiation in order to re-express then differentiate products and quotients that would otherwise be very difficult to differentiate.

1. EQ.9.1.A: Under what circumstances might you choose to use the technique of logarithmic differentiation?

ii. EU.9.2: The derivative of an exponential function is also an exponential function. The exponential function $y = e^x$ is the one exponential function that is equal to its own derivative.

1. EQ.9.2.A: Why might one expect the derivative of an exponential function to also be an exponential function?

10. *If time:* Unit Ten: Inverse Trigonometric Functions

a. Inverse Trigonometric Functions: Differentiation (LHE 5.6; AB; BC)

i. Review of differential rules for all six trigonometric functions (AB; BC)

ii. Use of implicit differentiation to find the derivative of an inverse trig function (AB; BC)

b. Inverse Trigonometric Functions: Integration (LHE 5.7)

c. *Enduring Understandings* and *Essential Questions* for Unit Ten

i. EU.10.1: An inverse trigonometric function outputs a single angle measure corresponding to a given value of a trigonometric ratio. In order for the inverse trigonometric function to output only one unique angle measure (a necessary property of all functions) for each inputted ratio value, certain restrictions are placed on the range of the function.

1. EQ.10.1.A: How is it possible for two different angle measures to generate the very same trigonometric ratio?

2. EQ.10.1.B: Why is it necessary to restrict the range of an inverse trigonometric function?

ii. EU.10.2: The derivative formulas for inverse trigonometric functions can be generated cleverly through the process of implicit differentiation and substitutions made based on (trigonometric) Pythagorean identities.

1. EQ.10.2.A: How are derivative formulas for inverse trigonometric functions derived?

11. Unit Eleven: Differential Equations (LHE 6.0)

a. Slope Fields and Euler's Method (LHE 6.1; AB; BC)

i. Geometric interpretation of differential equations via slope fields and the relationship between slope fields and solution curves for differential equations (AB; BC)

b. Differential Equations: Growth and Decay (LHE 6.2; AB; BC)

c. Separation of Variables and the Logistic Equation (LHE 6.3; AB; BC)

d. *Enduring Understandings* and *Essential Questions* for Unit Eleven

i. EU.11.1: A slope field shows short line segments where the slope of each segment is equal to the slope of a line tangent to a curve satisfying the conditions of a given differential equation.

1. EQ.11.1.A: What does a slope field reveal about the characteristics of a solution to the corresponding differential equation?

ii. EU.11.2: Following the guidelines established by a corresponding slope field, a sketch of a possible solution curve for a given differential equation can be drawn. These sketched curves can be likened unto the path a golf ball would follow should it land at a particular point on the sloped surface of a putting green.

1. EQ.11.2.A: How is a slope field similar to a putting green?

iii. EU.11.3: The rate of change of an exponential function is directly proportional to the value of the function. This relationship can be represented symbolically as $\frac{dy}{dx} = ky$.

1. EQ.11.3.A: What determines the instantaneous rate at which the output of an exponential function is changing?

12. Unit Twelve: Area and Volume (and Length and Work) (LHE 7.0)

a. Area of a Region Between Two Curves (LHE 7.1; AB; BC)

b. Volume: The Disk (and Washer) Method (LHE 7.2; AB; BC)

i. Volume of a solid with known cross sections (AB; BC)

c. *If Time:* Volume: The Shell Method (LHE 7.3)

d. *If Time:* Arc Length and Surfaces of Revolution (LHE 7.4)

e. *If Time:* Work (LHE 7.5)

f. *Enduring Understandings* and *Essential Questions* for Unit Twelve

i. EU.12.1: In the same way that a bound area is equal to the sum of the areas of infinitely many narrow rectangles, the volume of a bound solid is equal to the sum of the volumes of infinitely many extremely thin cross-sections.

1. EQ.12.1.A: How can definite integrals be used to determine the volume of a solid?

2. EQ.12.1.B: How is the shape and volume of an extremely thin cross-section determined?

ii. EU.12.2: The length of a curve is equal to the sum of the lengths of infinitely many short line segments.

1. EQ.12.2.A: How can the length of a curve be approximated and how is this approximation expressed in the form of a definite integral?