An Introduction to Inverse Functions

Supplement to Pre-Calculus Section 1.5 Inverse Functions

Let $f(x) = 3\left(\frac{x-4}{2} + 5\right)$. Shown below is an incomplete table showing various ordered pairs (inputs and corresponding outputs) for the function *f*. Said differently, the table below shows the coordinates of five of the infinitely many points that are on the graph of the function *f*.

x	$f(x)$	
10	24	
4	?	*For this portion of the table we are given an input and must find the corresponding output. The formula for the function f will help us turn these inputs into outputs.*
-12	?	
?	18	*For this portion of the table we are given an output and must find the corresponding input. To determine the input from which each output came we will need to "invert" the function f.*
?	3	

Part A | Confirming an Input and Output Pair

Let us confirm that the ordered pair $(10,24)$ represents a point on the graph of the function *f*.

$$f(10) = 3\left(\frac{10-4}{2} + 5\right) = 3\left(\frac{6}{2} + 5\right) = 3(3+5) = 3(8) = 24$$

Part B | Finding the Output for a Given Input

The table shows the ordered pair $(4, ?)$. We are given the input 4 but the output is unknown. We can put 4 into the function *f* to determine the corresponding output.

$$\begin{aligned} f(4) &= 3\left(\frac{4-4}{2} + 5\right) \\ &= 3\left(\frac{0}{2} + 5\right) \\ &= 3(0+5) \\ &= 3(5) \\ &= 15 \end{aligned}$$

Therefore $f(4) = 15$ and the point $(4,15)$ is on the graph.

The table shows the ordered pair $(-12, ?)$. We are given the input 4 but the output is unknown. We can put -12 into the function *f* to determine the corresponding output.

$$\begin{aligned} f(-12) &= 3\left(\frac{-12-4}{2} + 5\right) \\ &= 3\left(\frac{-16}{2} + 5\right) \\ &= 3(-8+5) \\ &= 3(-3) \\ &= -9 \end{aligned}$$

Therefore $f(-12) = 9$ and the point $(-12,-9)$ is on the graph.

Part C | Finding the Input that Produced a Given Output

Reconsider the following two ordered pairs appearing in the table for $f(x) = 3\left(\frac{x-4}{2} + 5\right)$.

x	$f(x)$
?	18
?	3

The table shows the ordered pair $(?, 18)$.
We are given the output 18 but the value of the input *x* that produced this output is unknown. All we know is that $f(x) = 18$ for some particular value of *x*.

$$18 = 3\left(\frac{x-4}{2} + 5\right)$$
$$6 = \frac{x-4}{2} + 5$$
$$1 = \frac{x-4}{2}$$
$$2 = x - 4$$
$$6 = x$$

Therefore $f(6) = 18$ and the point $(6,18)$ is on the graph. To describe the process of finding $x = 6$, we write $f^{-1}(18) = 6$. The expression $f^{-1}(18) = 6$ can be read as "*f inverse of 18 is 6*" and can be thought of a notation that communicates *the process of starting with the output 18 then undoing or inverting the operations of the function f in order to arrive at the input 6.*

The table shows the ordered pair $(?, 3)$.
We are given the output 3 but the value of the input x that produced this output is unknown. All we know is that $f(x) = 3$ for some particular value of *x*.

$$3 = 3\left(\frac{x-4}{2} + 5\right)$$
$$1 = \frac{x-4}{2} + 5$$
$$-4 = \frac{x-4}{2}$$
$$-8 = x - 4$$
$$-4 = x$$

Therefore $f(-4) = 3$ and the point $(3, -4)$ is on the graph. To describe the process of finding $x = 3$ we write $f^{-1}(-4) = 3$. The expression $f^{-1}(-4) = 3$ can be read as "*f inverse of -4 is 3*" and can be thought of a notation that communicates *the process of starting with the output -4 then undoing or inverting the operations of the function f in order to arrive at the input 3.*

Part D | Writing a Formula for the Inverse of a Given Function

For the function $f(x) = 3\left(\frac{x-4}{2} + 5\right)$, we have observed that:

To evaluate *f*:		To invert *f*:	
	Subtract 4. Divide by 2. Add 5. Multiply by 3.		Divide by 3. Subtract 5. Multiply 2. Add 4.

To confirm that the two processes are described correctly, select any number then perform upon it all four steps in the *evaluate* column on the left then take this result and perform upon it all four steps of the *inverse* column on the right. You should arrive at the same number you started with. The first set of four steps are inverted by the second set of four steps.

Each set of steps correspond to a function and a corresponding **inverse function** named f^{-1}.

Function: $f(x) = 3\left(\dfrac{x-4}{2} + 5\right)$ Inverse function: $f^{-1}(x) = 2\left(\dfrac{x}{3} - 5\right) + 4$

Since the table for *f* includes the input and output pair $(10,24)$ we know both that $f(10) = 24$ as well as $f^{-1}(24) = 10$. Let us confirm these statements with our two formulas:

Function:

$$\begin{aligned} f(10) &= 3\left(\frac{10-4}{2} + 5\right) \\ &= 3\left(\frac{6}{2} + 5\right) \\ &= 3(3+5) \\ &= 3(8) \\ &= 24 \end{aligned}$$

Inverse function:

$$\begin{aligned} f^{-1}(24) &= 2\left(\frac{24}{3} - 5\right) + 4 \\ &= 2(8-5) + 4 \\ &= 2(3) + 4 \\ &= 6 + 4 \\ &= 10 \end{aligned}$$

Part E | Proving that One Functions Inverts Another a Function

To prove that $f^{-1}(x) = 2\left(\frac{x}{3} - 5\right) + 4$ is the inverse of the function $f(x) = 3\left(\frac{x-4}{2} + 5\right)$, we can show that $f^{-1}(f(x)) = x$.

$$\begin{aligned}
f^{-1}(f(x)) &= f^{-1}\left(3\left(\frac{x-4}{2} + 5\right)\right) \\
&= 2\left(\frac{3\left(\frac{x-4}{2} + 5\right)}{3} - 5\right) + 4 \\
&= 2\left(\frac{x-4}{2} + 5 - 5\right) + 4 \\
&= 2\left(\frac{x-4}{2}\right) + 4 \\
&= x - 4 + 4 \\
&= x
\end{aligned}$$

Part F | Proving that Two Functions are Each Inverses of One Another

To prove that $f^{-1}(x) = 2\left(\frac{x}{3} - 5\right) + 4$ and $f(x) = 3\left(\frac{x-4}{2} + 5\right)$ are each inverses of one another, we can show that both $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

$$\begin{aligned}
f^{-1}(f(x)) &= f^{-1}\left(3\left(\frac{x-4}{2} + 5\right)\right) \\
&= 2\left(\frac{3\left(\frac{x-4}{2} + 5\right)}{3} - 5\right) + 4 \\
&= 2\left(\frac{x-4}{2} + 5 - 5\right) + 4 \\
&= 2\left(\frac{x-4}{2}\right) + 4 \\
&= x - 4 + 4 \\
&= x
\end{aligned}$$

$$\begin{aligned}
f(f^{-1}(x)) &= f\left(2\left(\frac{x}{3} - 5\right) + 4\right) \\
&= 3\left(\frac{2\left(\frac{x}{3} - 5\right) + 4 - 4}{2} + 5\right) \\
&= 3\left(\frac{2\left(\frac{x}{3} - 5\right)}{2} + 5\right) \\
&= 3\left(\frac{x}{3} - 5 + 5\right) \\
&= 3\left(\frac{x}{3}\right) \\
&= x
\end{aligned}$$

Therefore:

$x \rightarrow$ $\boxed{f}$ $\rightarrow$ $\boxed{f^{-1}}$ $\rightarrow x$

Therefore:

$x \rightarrow$ $\boxed{f^{-1}}$ $\rightarrow$ $\boxed{f}$ $\rightarrow x$

Part F | Writing a Formula for the Inverse of a Given Function *Revisited*

Let us now consider two different methods for writing the inverse of $g(x) = \sqrt[3]{\frac{2x+3}{5}}$.

Method One: "Undo" the Operations of the Given Function in Reverse Order

Starting with the last operation performed by g, we begin to undo or invert the operations of g in reverse order.

$$g(x) = \sqrt[3]{\frac{2x+3}{5}}$$

⇩

Multiply by 2.

⇩

Add 3.

⇩

Divide by 5.

⇩

Take the cube root.

Raise to the 3rd power.

⇩

Multiply by 5.

⇩

Subtract 3.

⇩

Divide by 2.

⇩

$$g^{-1}(x) = \frac{5x^3 - 3}{2}$$

Method Two: Solve for the Output "y"

$$y = \sqrt[3]{\frac{2x+3}{5}}$$

$$(y)^3 = \left(\sqrt[3]{\frac{2x+3}{5}}\right)^3$$

$$y^3 = \frac{2x+3}{5}$$

$$5(y^3) = 5\left(\frac{2x+3}{5}\right)$$

$$5y^3 = 2x + 3$$

$$5y^3 - 3 = 2x + 3 - 3$$

$$5y^3 - 3 = 2x$$

$$\frac{5y^3 - 3}{2} = \frac{2x}{2}$$

$$\frac{5y^3 - 3}{2} = x$$

We let y hold the place of an output that results from some particular input x. Then we solve for y.

$$g(x) = \text{"}y\text{"} = \sqrt[3]{\frac{2x+3}{5}}$$

The resulting formula gives us directions on how to undo the operations of the function g in order to find the input of g that produced the output y.

$$x = \frac{5y^3-3}{2}$$

We can rename this formula g^{-1} or $g^{-1}(y)$.

$$g^{-1}(y) = \frac{5y^3-3}{2}$$

Since the variable *"y"* is simply a placeholder, we can represent the very same formula using the placeholder x.

$$g^{-1}(x) = \frac{5x^3-3}{2}$$

Part G | Inverting a Given Function that is *Completely* Represented by a Table

Let the function h be represented *completely* by the table below. This means that there are exactly, or *only*, four input and output pairs for the function h.

x	$h(x)$
5	8
2	-4
0	7
-3	6

To create the table for the inverse function all we have to do is swap the columns. The *outputs* of the function h become the *inputs* of the function h^{-1}. The *inputs* of the function h become the *outputs* of the function h^{-1}.

x	$h(x)$
5	8
2	-4
0	7
-3	6

x	$h^{-1}(x)$
8	5
-4	2
7	0
6	-3

The following four statement show that $h^{-1}(h(x)) = x$ and $h(h^{-1}(x)) = x$, thereby confirming that we have successfully created tables for functions that are inverses of one another:

$$\boxed{h^{-1}(h(x)) = x}$$

$$h^{-1}(h(5)) = h^{-1}(8) = 5$$
$$h^{-1}(h(2)) = h^{-1}(-4) = 2$$
$$h^{-1}(h(0)) = h^{-1}(7) = 0$$
$$h^{-1}(h(-3)) = h^{-1}(6) = -3$$

$$\boxed{h(h^{-1}(x)) = x}$$

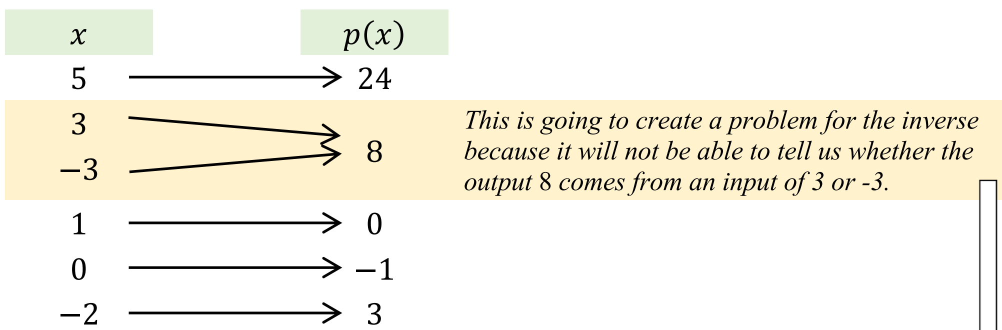
$$h(h^{-1}(8)) = h(5) = 8$$
$$h(h^{-1}(-4)) = h(2) = -4$$
$$h(h^{-1}(7)) = h(0) = 7$$
$$h(h^{-1}(6)) = h(-3) = 6$$

Part H | Functions Represented by Tables may *Not* actually have an Inverse Function

Let the function *p* be represented *completely* by the table below. This means that there are exactly, or *only*, six input and output pairs for the function *p*.

x	$p(x)$
5	24
3	8
1	0
0	-1
-2	3
-3	8

Let us take a look at the same function *p* represented by a mapping.



Suppose we apply the same method we learned in Part G. In an attempt to create the table for the inverse function, we swap the columns.

x	$p(x)$
5	24
3	8
1	0
0	-1
-2	3
-3	8

x	$p^{-1}(x)$
24	5
8	$3 \text{ or } -3?$
0	1
-1	0
3	-2

Although $p(x)$ is a function, we cannot call $p^{-1}(x)$ a "function" because there is a number that we can put into $p^{-1}(x)$ that does not produce just one exact value.

Part I | Only One-to-One Functions are Invertible

Consider the four different tables of input and output pairs.

x	$r(x)$
1	7
2	11
3	15
4	19

x	$r^{-1}(x)$
7	1
11	2
15	3
19	4

x	$s(x)$
1	2
2	6
3	12
4	20

x	$s^{-1}(x)$
2	1
6	2
12	3
20	4

x	$v(x)$
1	4
2	5
2	7
4	6

Recall that for a relationship between inputs and outputs to be a function it must be the case that every input is paired with exactly one output. The table for v cannot be a function because the input 2 is paired with two outputs, 5 and 7. If v is not a function then it does not make any sense to consider the possibility of an inverse function for v.

x	$w(x)$
1	4
2	1
3	0
4	1

The table for w does satisfy the requirements that let us call w a function. However, we cannot create an inverse function for w because our inverse function will not "know what to do" with 1. It will not be able to tell us whether 1 was produced by w from 2 or 3.

The tables above demonstrate that in order for a given function to have an inverse function, every input of the given function must be paired with exactly one output *and* every output of the given function must be paired with exactly one input. We call a function satisfying these conditions a **one-to-one** function. In order for a function to be invertible it must be one-to-one.

Part J | Determining Whether or Not a Given Formula for a Function is Invertible

Let us compare and contrast two different functions. The first function *f* is a one-to-one function and is invertible. The second function *g* is *not* a one-to-one function and is therefore *not* invertible.

Given function f:	*Given function g:*
$f(x) = (x-5)^3$	$g(x) = x^2 + 7$

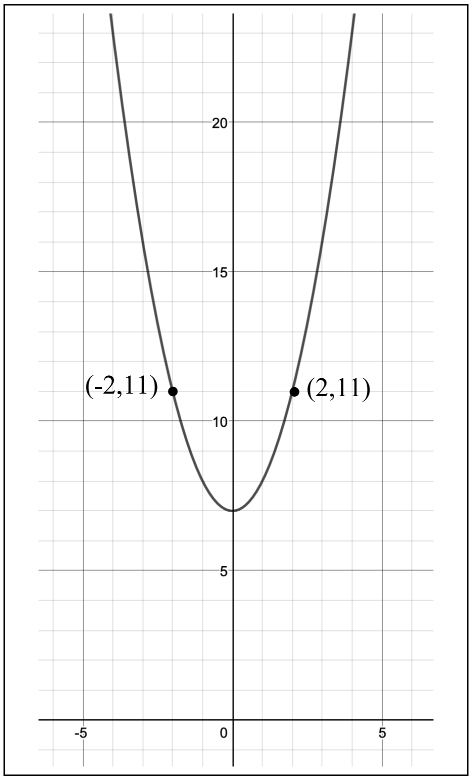
Let's attempt Method Two: Solve for the Output "y" to find an inverse function.

$y = (x-5)^3$	$y = x^2 + 7$
$\sqrt[3]{y} = \sqrt[3]{(x-5)^3}$	$y - 7 = x^2 + 7 - 7$
$\sqrt[3]{y} = x - 5$	$y - 7 = x^2$
$\sqrt[3]{y} + 5 = x - 5 + 5$	*Take the square root of both sides.*
$\sqrt[3]{y} + 5 = x$	*Remember to include a ± sign.*
Flip the equation.	$\pm\sqrt{y-7} = \sqrt{x^2}$
$x = \sqrt[3]{y} + 5$	$\pm\sqrt{y-7} = x$
Name this the inverse function.	*Flip the equation.*
$f^{-1}(y) = \sqrt[3]{y} + 5$	$x = \pm\sqrt{y-7}$
Replace the placeholder y with x.	*Notice that this cannot be a function because it generates two different values for most numbers you put into the formula.*
$f^{-1}(x) = \sqrt[3]{x} + 5$	

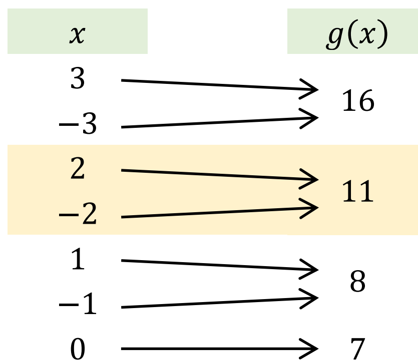
To further demonstrate that the function $g(x) = x^2 + 7$ does not have an inverse function notice that $g(4) = (4)^2 + 7 = 23$ and $g(-4) = (-4)^2 + 7 = 23$. The function *g* is *not* one-to-one. We cannot create an inverse function for *g* that will be able to tell us precisely where the 23 came from. An inverse function would be "confused" as to whether or not the 23 comes from a 4 or -4.

Part K | Explicit Domain Restrictions Can Make a Function Invertible

Let’s revisit the function $g(x) = x^2 + 7$. Consider a graph of the function as well as (an incomplete) table of certain input and output pairs and a mapping for those selected pairs.



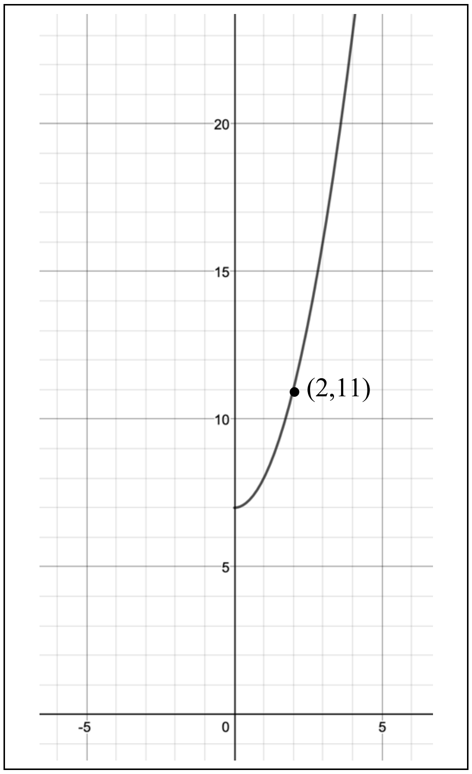
x	$g(x)$
3	16
2	11
1	8
0	7
-1	8
-2	11
-3	16



The implied domain of this function is $(-\infty, \infty)$ or said differently, the domain of the function is all real numbers. The corresponding range of the function is $[7, \infty)$ or said differently, all real numbers greater than or equal to 7. This graph is clearly not one-to-one and is therefore not invertible. For example, the inputs 2 and -2 both produce an output of 11. We cannot create an inverse function that will be able to tell us precisely where the 11 came from. An inverse function would be “confused” as to whether or not the 11 comes from a 2 or -2.

But suppose that the domain of a function is explicitly restricted…

Suppose that the function $h(x) = x^2 + 7$ has a domain that is explicitly restricted to $[0, \infty)$, or said differently a domain restricted to all real numbers greater than or equal to zero. Consequently the corresponding range of the function is still $[7, \infty)$. This makes the graph as well as the table and mapping of points all look very different than the previous function *g*.



x	$h(x)$
3	16
2	11
1	8
0	7

Notice that the table and mapping show that the function h is a one-to-one function.

x		$h(x)$
3	$\longrightarrow$	16
2	$\longrightarrow$	11
1	$\longrightarrow$	8
0	$\longrightarrow$	7

Does the function *h* have an inverse function? Yes, it does. Let's use *Method Two: Solve for the Output "y"* to find an inverse function.

$$y = x^2 + 7$$
$$y - 7 = x^2 + 7 - 7$$
$$y - 7 = x^2$$
$$\sqrt{y-7} = \sqrt{x^2}$$
$$\sqrt{y-7} = x$$

Now when we take the square root of both sides we will not include the $\pm$ sign because the explicitly stated domain of h makes it clear that the input x of the function h cannot be negative.

$$x = \sqrt{y-7}$$

Name this the inverse function.

$$h^{-1}(y) = \sqrt{y-7}$$

Replace the placeholder y with x.

$$h^{-1}(x) = \sqrt{x-7}$$

This inverse function can now tell us exactly how the function *h* produces 11. According to the formula $h^{-1}(11) = \sqrt{11-7} = \sqrt{4} = 2$, and only 2.